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Self-Projective Modules with π-Injective Factor Modules

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By an old theorem of Barbara Osofsky, a ring R is left semisimple if and only if every cyclic left R-module is injective. This result was the motivation for studying rings whose (proper) cyclic modules are self-injective or π-injective (quasi-continuous) (e.g., Goel and Jain [2], Jain and Mohamed [6], Klatt and Levy [7], Koehler [8], and Osofsky and Smith [9]). Assuming certain decomposition properties, the structure of projective modules with π-injective factor modules was investigated in Tuganbaev [10].

In this note we provide a structure theorem for finitely generated, self-projective modules whose factor modules are π-injective: These modules have a decomposition $M = M_0 \oplus M_1 \oplus M_2$ with fully invariant submodules M_i, where M_0 is semisimple, M_1 is a direct sum of uniserial modules whose endomorphism ring is a division ring, and M_2 is a direct sum of fully invariant uniform submodules whose endomorphism rings are not division rings (Theorem 2.2).

It is also shown that a finitely generated, self-projective local module M, whose factor modules are continuous, is uniserial, its M-generated submodules are fully invariant, and its endomorphism ring is left uniserial and left duo (Proposition 1.2).

Generalizing an observation known for commutative rings (see [7]) we derive in Proposition 1.3 that a cyclic uniserial self-injective module has a right linearly compact endomorphism ring.

Finitely generated, self-projective indecomposable modules with self-injective factor modules are uniserial and their endomorphism rings are left

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uniserial left duo rings and are linearly compact and uniserial on the right. Also, in this case every finitely M-generated module is serial and all indecomposable injective modules in $\sigma[M]$ are uniserial (Theorem 2.1). This transfers part of the Theorem in Gill [1] from local commutative rings to local modules.

1. PRELIMINARIES

Let R be an associative ring with unit and R-Mod the category of unital left R-modules. Morphisms are written on the opposite side of the scalars.

Throughout the paper M will denote a module in R-Mod. A left R-module is called *M-generated* if it is a homomorphic image of a direct sum of copies of M. By $\sigma[M]$ we denote the full subcategory of R-Mod whose objects are submodules of M-generated modules.

M is called *self-projective* or *quasi-projective* if it is M-projective; it is called *self-injective* or *quasi-injective* if it is M-injective. M is said to be a *self-generator* if every submodule of M is M-generated. $\ker(f)$ denotes the kernel of a module homomorphism. For other basic definitions see [11].

An R-module in which (intersection) complement submodules are direct summands is called a *CS module*. These are modules for which every submodule is essential in a direct summand.

An R-module M is called *π-injective* or *quasi-continuous* if, for any submodules $U, V \subset M$ with $U \cap V = 0$, the canonical monomorphism $M \to M/U \oplus M/V$ splits. This is equivalent to demanding that M be a CS module and that, for any direct summands U and V of M with $U \cap V = 0$, the sum $U + V$ be a direct summand of M (e.g., [11, 41.20 and 41.21]).

M is said to be *direct-injective* if every submodule isomorphic to a direct summand is a direct summand in M. In particular, the endomorphism ring of a direct-injective uniform module is local (e.g., [11, 41.22]).

A module which is π-injective and direct-injective is called *continuous*.

Rings whose cyclic modules have any of the above properties have been studied by various authors (e.g., [2, 6, 7, 8, 9]). Here we investigate more generally self-projective modules whose factor modules have these properties. Results for rings are obtained as corollaries.

1.1. PROPOSITION. *Consider an R-module $M = M_1 \oplus \cdots \oplus M_k$ with uniform modules M_i and assume every factor module of M to be π-injective. Then:*

(1) *Every non-zero $f \in \mathrm{Hom}(M_i, M_j)$ with $i \neq j$ is an epimorphism.*

(2) *If M_i is M_j-projective, then f is an isomorphism.*

(3) *If not every $h \in \mathrm{End}(M_j)$ is epic, then $\mathrm{Hom}(M_i, M_j) = 0$ for $i \neq j$.*

Proof. (1), (2) are proved in Lemma 8 of [10] and in Lemma D of [9].

(3) Assume $h \in \operatorname{End}(M_j)$ is not epic and there is a non-zero $f \in \operatorname{Hom}(M_i, M_j)$ which must be epic by (1). Then $M_i \oplus M_j h$ is M-cyclic and π-injective and by [11, 41.20], $M_j h$ is M_i-injective. Since we have an epimorphism $f\colon M_i \to M_j$ the module $M_j h$ is also M_j-injective (e.g., [11, 16.2]) and hence a direct summand in M_j, a contradiction.

A module is called *uniserial* if its submodules are linearly ordered by inclusion. *Serial modules* are direct sums of uniserial modules (see [11, Sect. 55]). A ring S is called *left* (*right*) *duo* if every left (right) ideal is a two-sided ideal.

A module is said to be *local* if it has a maximal submodule which is superfluous. A finitely generated, self-projective module is local if and only if it has a local endomorphism ring or—equivalently—all its factor modules are indecomposable (e.g., [11, 19.7]).

Generalizing the notion of a flat module we call a right S-module M_S *weakly flat* provided the functor $M \otimes_S -$ is exact with respect to all exact sequences $0 \to I \to S$ with cyclic left ideals $I \subset S$. It is well-known that a left R-module M is flat over $S = \operatorname{End}({}_RM)$ if and only if the kernels of morphisms $M^k \to M^l$, $k, l \in \mathbb{N}$, are M-generated (e.g., [11, 15.9]). The same proof shows that M_S is weakly flat if and only if the kernels of endomorphisms of M are M-generated.

1.2. PROPOSITION. *Let M be a finitely generated, self-projective, and local R-module and* $S = \operatorname{End}(M)$.

(1) *If every factor module of M is a CS module, then M is uniserial and ${}_SS$ is uniserial.*

(2) *If every factor module of M is continuous, then the M-generated submodules of M are fully invariant and S is a left duo ring.*

(3) *If every factor module of M is continuous and M_S is a weakly flat S-module, then the Jacobson radical of S is a nil ideal.*

Proof. (1) Factor modules of M are indecomposable and CS and hence uniform. This implies that M is uniserial (see [11, 55.1]) and ${}_SS$ is uniserial (see [11, 55.2]).

(2) Modifying the proof of Proposition 2.2 in [6] we show that, for every $f \in S$, the R-submodule Mf is also an S-module of M:

First consider an isomorphism $g \in S$. Suppose $Mfg \not\subset Mf$. By (1), this implies that $Mf \subset Mfg$. Since $Mf \simeq Mfg$ is continuous, Mf is a direct summand of the indecomposable module Mfg and hence $Mf = Mfg$, a contradiction.

If $g \in S$ is not an isomorphism, then $1-g$ is an isomorphism and we know from above that $Mf(1-g) \subset Mf$. Since for every $m \in M$,

$$mfg = mf - mf + mfg = mf - mf(1-g),$$

we get $Mfg \subset Mf$.

Under the given conditions, for every left ideal $I \subset S$, $I = \mathrm{Hom}(M, MI)$ (e.g., [11, 18.4]). By our above result $Mf = MfS$ for $f \in S$ and hence

$$Sf = \mathrm{Hom}(M, MSf) = \mathrm{Hom}(M, MSfS) = SfS;$$

i.e., S is a left duo ring.

(3) To show that S is continuous, we prove that, for any idempotent $e \in S$, every monomorphism $\gamma: Se \to S$ splits: Tensoring with $M_S \otimes$ — we get a monomorphism

$$Me \simeq M \otimes Se \xrightarrow{\mathrm{id} \otimes \gamma} M \otimes S \simeq M,$$

which splits since M is continuous. From this we derive that γ also splits.

For a left ideal $I \subset S$, MI is fully invariant and the exact sequence

$$0 \to \mathrm{Hom}(M, MI) \to \mathrm{End}(M) \to \mathrm{Hom}(M, M/MI) \to 0$$

yields $S/I \simeq \mathrm{End}(M/MI)$. M/MI is again a self-projective continuous module and its endomorphism ring is continuous by our preceding observation.

We have seen that S is a ring whose cyclic left modules are continuous. By Proposition 2.2 of [6] or Corollary 9 of [9], such a ring has nil Jacobson radical.

In Klatt and Levy [7] the relationship between injectivity and linear compactness for commutative valuation rings was pointed out. Theorem 2.3 of [7] is a special case of the following more general observation. For $M = R$ our condition (ii) corresponds to a double annihilator condition.

1.3. PROPOSITION. *For a finitely generated uniserial R-module M with $S = \mathrm{End}(M)$ the following properties are equivalent*:

(a) *M is self-injective*;

(b) *S_S is linearly compact and*

(i) *$\mathrm{Hom}(-, M)$ is exact with respect to exact sequences $0 \to K \to M$ with K finitely generated.*

In case M is a self-generator, (i) *is equivalent to*

(ii) *For every $f \in S$, $fS = \mathrm{Hom}(M/\mathrm{ker}(f), M)$.*

Proof. (a)$\Rightarrow$(b) Since M is self-injective we know from [11, 55.2] that S_S is uniserial. To prove that S_S is linearly compact we show that, for every inverse family of right ideals $\{I_\alpha\}_A$, the canonical morphism $S \to \varprojlim S/I_\alpha$ is epic (e.g., [11, 29.7]). Since S_S is uniserial it is easily verified that every right ideal is the intersection of the cyclic right ideals containing it. Hence we may assume that the ideals I_α are (finite intersections of) cyclic right ideals.

M being self-injective, for every $f \in S$,

$$fS = \mathrm{Hom}(M/\ker(f), M) \qquad \text{and} \qquad S/fS \simeq \mathrm{Hom}(\ker(f), M).$$

Also, $\ker(f)$ is M-reflexive, i.e., $\ker(f) \simeq \ker(f))^{**}$, where $(—)^*$ denotes the contravariant functors $\mathrm{Hom}(—, {}_RM)$ and $\mathrm{Hom}(—, M_S)$. Hence S/fS is also M-reflexive (e.g., [11. 47.4]).

The inverse family $\{S/I_\alpha\}_A$ yields a direct family of submodules $(S/I_\alpha)^* \subset S^* \simeq M$ and the functor $\mathrm{Hom}(—, {}_RM)$ leads to the exact sequence

$$S^{**} \to \mathrm{Hom}(\varinjlim\,(S/I_\alpha)^*, M) \to 0.$$

The S/I_α are reflexive and $\mathrm{Hom}(—, M)$ converts direct limits into inverse limits. Therefore we have the commutative diagram with exact lower row

$$\begin{tikzcd}
S \arrow[r] \arrow[d, "\simeq"] & \varprojlim\,(S/I_\alpha) \arrow[r] \arrow[d, "\simeq"] & 0 \\
S^{**} \arrow[r] & \varprojlim\,(S/I_\alpha)^{**} \arrow[r] & 0.
\end{tikzcd}$$

From this we see that the upper row is exact and S_S is linearly compact.

It follows from [11, 28.2] that under the given assumptions the properties (i) and (ii) are equivalent.

(b)$\Rightarrow$(a) Let $0 \to L \to M$ be an exact sequence and write $L = \varinjlim L_\alpha$ as a direct limit of cyclic submodules L_α. Since $\mathrm{Hom}(—, M)$ is exact with respect to the exact sequences $0 \to L_\alpha \to M$ and $\mathrm{Hom}(—, M)$ converts direct limits into inverse limits we have the first row exact in the diagram

$$\begin{tikzcd}
\mathrm{Hom}(M, M) \arrow[r] \arrow[d, "\simeq"] & \varprojlim \mathrm{Hom}(L_\alpha, M) \arrow[r] \arrow[d, "\simeq"] & 0 \\
\mathrm{Hom}(M, M) \arrow[r] & \mathrm{Hom}(L, M) \arrow[r] & 0.
\end{tikzcd}$$

This implies that the second row is also exact and M is self-injective.

2. STRUCTURE THEOREMS

The next result extends some of the implications of the Theorem in [1] from commutative local rings to local modules:

2.1. THEOREM. *Let M be a finitely generated, self-projective, and indecomposable R-module whose factor modules are self-injective. Then*:

(1) *M is uniserial and every M-generated submodule of M is fully invariant.*

(2) *$S = \mathrm{End}(M)$ is a left uniserial left duo ring and is linearly compact and uniserial on the right.*

(3) *Every indecomposable injective module in $\sigma[M]$ is uniserial.*

(4) *Every finitely M-generated module is serial.*

(5) *If M_S is weakly flat, then S also is a right duo ring.*

Proof. (1), (2) The properties stated for M are established in the preceding propositions and we have also seen that S is left duo and uniserial as well as right uniserial and linearly compact.

(3) Let U be an indecomposable injective module in $\sigma[M]$. Since U is M-generated it is enough to show that the M-generated submodules of U are uniserial: Consider $f, g \in \mathrm{Hom}(M, U)$ and assume $\ker(f) \subset \ker(g)$. Then Mg is isomorphic to a factor module of Mf. Since Mf is self-injective it is also Mg-injective and hence Mf is a direct summand of $Mf + Mg \subset U$. But U is uniform and hence $Mf + Mg = Mf$ is uniserial. This implies that U is uniserial.

(4) By [11, 55.10], property (3) is equivalent to (4).

(5) If M_S is weakly flat, then $\ker(f)$ is M-generated for every $f \in S$ (see remarks before Proposition 1.2) and hence is fully invariant by (1). Hence for any $g \in S$ we know $\ker(gf) \subset \ker(f)$ and

$$gfS = \mathrm{Hom}(M/\ker(gf), M) \subset \mathrm{Hom}(M/\ker(f), M) = fS.$$

From this we see that fS is a two-sided ideal.

Another possibility for proving (5) is to recall from Proposition 1.2 that, under the given conditions, the Jacobson radical of S is nil. It has been shown in the proof of Proposition 3.1 of [5] and also in a remark preceding Corollary 10 of [9] that a right uniserial ring with nil Jacobson radical is right duo.

In a decomposition of an R-module $N = N_1 \oplus N_2$, the R-submodules N_1, N_2 are fully invariant if and only if there are no non-zero morphisms between them. Combining this observation with our preceding results we get:

2.2. THEOREM. *Let M be a finitely generated, self-projective R-module whose factor modules are π-injective. Then there is a decomposition*

$$M = M_0 \oplus M_1 \oplus M_2$$

with fully invariant submodules M_0, M_1, M_2 and

M_0 is a semisimple module,

$M_1 \simeq N_1^{k_1} \oplus \cdots \oplus N_r^{k_r}$ with fully invariant summands $N_i^{k_i}$, N_i non-simple uniserial (and self-injective if $2 \leqslant k_i$), and $\mathrm{End}(N_i)$ a division ring, and

$M_2 = U_1 \oplus \cdots \oplus U_k$ with fully invariant uniform modules U_i with $\mathrm{End}(U_i)$ not a division ring.

$\mathrm{End}(M) = \mathrm{End}(M_0) \times \mathrm{End}(M_1) \times \mathrm{End}(M_2)$ *with* $\mathrm{End}(M_0)$, $\mathrm{End}(M_1)$ *semisimple Artinian rings and* $\mathrm{End}(M_2) = \mathrm{End}(U_1) \times \cdots \times \mathrm{End}(U_k)$.

(1) *If M is a self-generator, then M_1 is zero.*

(2) *M is semiperfect in $\sigma[M]$ if and only if all the U_i are uniserial.*

(3) *If all factor modules of M are continuous, then the U_i are uniserial, the M_2-generated submodules of M_2 are fully invariant, and* $\mathrm{End}(M_2)$ *is a left serial left duo ring. If, in addition, M_S is weakly flat, then* $\mathrm{Jac}(S)$ *is nil.*

(4) *If every factor module of M is self-injective, then* $\mathrm{End}(M_2)$ *is right uniserial and linearly compact. If, in addition, M_S is weakly flat, then* $\mathrm{End}(M_2)$ *is also a right duo ring.*

Proof. By Corollary 1.4 in [4], a finitely generated, self-projective R-module M whose factor modules are CS has a decomposition as a direct sum of uniform modules. Since all summands are M-projective they have endomorphisms which are not epic if and only if the endomorphism ring is not a division ring.

In M_0 we collect all simple summands.

In M_1 we put all uniform summands whose endomorphism rings are division rings. By Proposition 1.2, these are uniserial. They are isomorphic or there exists no non-trivial morphisms between them by Proposition 1.1.

M_2 consists of the remaining summands. Their endomorphism rings are not division rings and hence, by Proposition 1.1, there are no non-zero morphisms between them; i.e., they are fully invariant.

Again by Proposition 1.1, there are no non-zero morphisms between M_0, M_1, and M_2. Hence we have a decomposition of M into fully invariant submodules which implies the ring theoretic decomposition of $\mathrm{End}(M)$ in the given form.

(1) If M is a self-generator, then also the N_i are self-generators. Since their endomorphism rings are division rings they must be simple.

(2) If M is semiperfect in $\sigma[M]$ the indecomposable summands have local endomorphism rings (e.g., [11, 42.4]) and the assertion follows from Proposition 1.2. On the other hand, cyclic uniserial M-projective modules are semiperfect in $\sigma[M]$ and a finite direct sum of semiperfect modules is again semiperfect in $\sigma[M]$.

(3) follows from Proposition 1.2 and the decomposition of M_2 given above.

(4) is a consequence of Theorem 2.1.

The situations (2), (3), (4) described in the above theorem extends the essential parts of the following results in the literature from rings to finitely generated self-projective modules: Theorem 2.4 in Goel and Jain [2] and Proposition 3 in Osofsky and Smith [9], Theorem 2.8 in Jain and Mohamed [6] and Corollary 9 in [9], and the Main Theorem in Koehler [8] (also Corollary 10 in [9]).

In Lemma 8 of [10], Tuganbaev investigated the structure of a projective module with π-injective factor modules under the assumption of certain decomposition properties. These conditions imply that the module is finitely generated. Hence Theorem 2.2 also includes his results on this topic and in addition it reveals which decompositions may occur.

In Theorem 2.2 the conditions *finitely generated* and *self-projective* for M are needed first of all to show that M has finite uniform dimension. If we assume the existence of certain decompositions (compare [10]) or assume some other (local) finiteness conditions, similar structure theorems may be obtained for modules with (π-) injective factor modules which are neither finitely generated nor self-projective (e.g, $\mathbb{Q}/\mathbb{Z}$ as $\mathbb{Z}$-module).

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